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CONFORMAL MAPPING PROCEDURE FOR TRANSIENT AND STEADY-STATE TWO-DIMENSIONAL SOLIDIFICATION

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TECHNICAL PAPER proposed for presentation at
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Abstract

A conformal mapping method was developed for analyzing two-dimensional transient and steady-state solidification problems. The method was applied to the solidification which takes place on a cold plate of finite width immersed in a flowing liquid and to the solidification inside of a cooled rectangular channel which contains a warm flowing liquid. The transient and steady-state shapes of the frozen regions are investigated.

INTRODUCTION

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A method was developed for solving two-dimensional transient and steady state solidification problems. The method is applicable to the case where a warm liquid at a temperature above its equilibrium freezing point flows steadily over a surface which is cooled below the freezing point. This may occur, for example, inside the conduits of certain rectangular heat exchangers. The method is applied to two specific cases which are illustrated in figures 1 and 2. The first of these consists of the frozen region formed on a cooled plate immersed in a warm flowing liquid. The second consists of the frozen region which forms inside of a rectangular channel when the channel walls are maintained at a constant temperature which is below the freezing temperature of the liquid.

In general, the flowing liquid supplies energy by convection to the solid-liquid interface. The shape of the frozen region adjusts so that this energy along with the latent heat of fusion, which is released in the transient situation, can be removed by conduction through the frozen region to the cold boundaries. In the transient situation there is also internal energy removed as the frozen material is cooled below its freezing point. In the present analysis this energy of subcooling is neglected. This assumption is a reasonable one to make because in a great many solidification problems, the latent heat released at the solid-liquid interface is much greater than the energy of subcooling. It is also assumed that the solid-liquid interface is at the equilibrium freezing temperature.

SYMBOLS

A dimensionless half width, $(ha/k)[(t_l - t_f)/(t_f - t_w)]$

A_n time dependent coefficients in mapping

a half width of plate; half width of long side of channel

B dimensionless half width, $(hb/k)[(t_l - t_f)/(t_f - t_w)]$

b	half width of short side of channel
c,d	intermediate mapping parameters
E	complete elliptic integral of the second kind
F	normal elliptic integral of the first kind
h	heat transfer coefficient from flowing liquid to frozen interface
$I(c_1,c_2)$	$\int_{c_1}^{\infty}\left[\frac{t+\sqrt{t^2-1}}{(t^2-1)(t+c_2)(t-c_1)}\right]^{1/2} dt$
I_Θ	frozen region in Z-plane
J_Θ	frozen region in W-plane
K	complete elliptic integral of the first kind
K_n	$\int_0^{\pi/2} \frac{\cos 2n\omega}{\sqrt{1-\beta^2 \sin^2\omega}}\, d\omega, \quad n = 0, 1, 2, \ldots$
k	thermal conductivity of solidified material
L	frozen region in ζ plane
M	$(\beta A_0 - A)/\ln \sqrt{1-\beta^2}$
$\hat{n}$	outward normal
Q	heat flow rate through frozen layer per unit length
$\vec{r}_s$	position vector to frozen interface
T	dimensionless temperature, $(t - t_w)/(t_f - t_w)$
t	temperature
t_f	freezing temperature
t_l	liquid temperature
t_w	surface temperature of cold plate or channel wall
W	analytic function, $-T + i\Psi$
X,Y	dimensionless coordinates, $(x/a)A$, $(y/a)A$
x,y	Cartesian coordinates in physical plane
Z	dimensionless complex physical plane, $X + iY$
z	physical plane, $x + iy$
α_n	time dependent coefficients in mapping equation

β	time dependent parameter in mapping
$\beta_{initial}$	initial value of β
Γ	frozen region in Ω-plane
γ_n	defined by $\sum_{k=0}^{n} \gamma_k \gamma_{n-k} = \alpha_n$; $n = 0, 1, 2, \ldots$
δ_{ij}	Kronecker delta
ζ	quantity defined as $[-(\partial T/\partial X) + i(\partial T/\partial Y)]^{-1}$
Θ	dimensionless time $(h^2/k\rho\lambda)\left[(t_l - t_f)^2/(t_f - t_w)\right]\theta$
θ	time
Ψ	imaginary part of W
λ	latent heat of fusion
ξ	parametric variable
ρ	density of solidified material
Ω	intermediate mapping plane
ω	argument in Ω-plane

Subscript:

s	on frozen interface

Superscript:

ss	steady state

GENERAL ANALYSIS

According to the model adopted the solid-liquid interface is at a constant (with both time and position) temperature t_f. Since the shape of this interface is unknown it is necessary to specify an additional boundary condition along it. Assume that the heat transfer coefficient h at the solid liquid interface is constant. Then at steady state the heat flux into the frozen region $k\hat{n}\cdot\nabla t$ is uniform along the interface and equal to the convective heat supplied by the flowing liquid $h(t_l - t_f)$. During the transient, however, the rate of freezing is in general nonuniform along the interface and the heat flux entering the frozen material is an unknown function of position and time which is determined from the condition

$$k\hat{n}\cdot\nabla t - h(t_l - t_f) = \rho\lambda\hat{n}\cdot\partial\vec{r}_s/\partial\theta \tag{1}$$

that results from applying an energy balance at the solid-liquid interface.

It is convenient to introduce a nondimensional temperature T defined by

$T = (t - t_w)/(t_f - t_w)$. All lengths are nondimensionalized by $k/h(t_f - t_w)/(t_l - t_f)$ and the time is made nondimensional by $(k\rho\lambda/h^2)[(t_f - t_w)/(t_l - t_f)^2]$. The dimensional quantities are denoted by lower case letters and the dimensionless quantities are denoted by the corresponding capital letters.

With the subcooling neglected the heat flow in the solidified region is governed by the two-dimensional Laplace's equation (in normalized coordinates), that is, at each instant of time the temperature T within the solidified region is a harmonic function of position. Hence, let Ψ be the harmonic function which is conjugate to $-T$. Then the complex function $W = -T + i\Psi$ is a function of time and at each fixed instant of time is an analytic function of the complex variable $Z = X + iY$. In view of this we use the notation $\partial W/\partial Z$ to denote the ordinary derivative of the analytic function W with respect to the complex variable Z.

It is convenient to introduce the complex variable ζ defined by

$$\zeta = \frac{\partial Z}{\partial W} \tag{2}$$

Clearly, at each instant of time ζ is an analytic function of the complex variable Z. The function ζ is related to the reciprocal of the complex temperature gradient in the frozen region since

$$\frac{1}{\zeta} = \frac{\partial W}{\partial Z} = -\frac{\partial T}{\partial X} + i\,\frac{\partial T}{\partial Y} \tag{3}$$

At each instant of time the functions W and ζ may be thought of as mappings of the instantaneous frozen region I_Θ in the physical plane into a region J_Θ in the complex W-plane and a region L_Θ in the complex ζ-plane, respectively. Specifying boundary conditions on the functions W and ζ along the complete boundary of I_Θ is equivalent to specifying the shapes of the regions J_Θ and L_Θ. Once the shapes of J_Θ and L_Θ are known, it is possible, at least in principle, to introduce an intermediate variable Ω, choose a certain region Γ in the Ω-plane and then to find the functions $\Omega \to W$ and $\Omega \to \zeta$ which map Γ into J_Θ and L_Θ, respectively. When these functions are known the integral

$$Z = \int \zeta\,\frac{\partial W}{\partial \Omega}\,d\Omega + \text{function of time}, \tag{4}$$

obtained by integrating equation (2), can be evaluated. The integral and the known function $\Omega \to W$ relate W to Z through the parametric variable Ω. Hence, at each instant of time, the temperature is known at each point of the region I_Θ in the physical plane. Since the solid-liquid interface corresponds to a particular isotherm this correspondence determines the shape of the frozen region in the physical plane. Thus, once the shapes of the regions J_Θ and L_Θ are known the solution to the problem can be found.

Some important differences between the steady state and transient cases should be noted. For the steady state cases the regions in the ζ and W planes can be determined directly from the boundary conditions. This is because the uniformity of the heat flux at the solid-liquid interface implies that $|\zeta|$ is constant there. In the transient freezing problem the shapes of the regions J_Θ and L_Θ change with time, however, it is convenient to fix Γ so that its shape and size are independent of time. Also, part of the boundary of L_Θ is unknown and must be determined by solving a nonlinear equation. If the region Γ is suitably chosen; however, the mapping $\Omega \to \zeta$

can be represented by a Taylor series with real coefficients which are functions of time only. These coefficients can be determined by substituting this series into the boundary condition (2).

ANALYSIS FOR FREEZING ON PLATE

The method is best illustrated by considering the situation depicted in figure 1. The cross section of the frozen layer configuration is shown in nondimensional coordinates in figure 3. The nondimensional boundary conditions are all shown on the figure except for the one given by equation (1). The boundary conditions expressed in terms of the complex variables W and ζ are

$$\left.\begin{aligned} \mathcal{R}e\, W(Z,\Theta) &= -1; \; Z \in \widehat{FAB} \\ \mathcal{R}e\, W(Z,\Theta) &= 0; \; Z \in \widehat{EDC} \end{aligned}\right\} \tag{5}$$

$$\left.\begin{aligned} \mathcal{I}m W(Z,\Theta) &= \text{function of time}; \; Z \in \widehat{FE} \\ \mathcal{I}m W(Z,\Theta) &= \text{function of time}; \; Z \in \widehat{CB} \end{aligned}\right\} \tag{6}$$

$$\mathcal{I}m \zeta(Z,\Theta) = 0 \quad \begin{cases} Z \in \widehat{CB} \\ Z \in \widehat{EF} \end{cases} \tag{7}$$

$$\mathcal{R}e\, \zeta(Z,\Theta) = 0 \qquad Z \in \widehat{EDC} \tag{8}$$

$$1 + \mathcal{R}e\left[\frac{dZ_s}{d\Theta}\, \overline{\zeta(Z,\Theta)}\right] = |\zeta(Z,\Theta)| \qquad Z \in \widehat{FAB} \tag{9}$$

Equations (5) and (6) show that at each instant of time the region I_Θ in the physical plane maps into the rectangular region J_Θ as indicated in figure 4. The height of the rectangle J_Θ varies with time and must be determined from the solution to the problem. Equations (7) and (8) show that the region I_Θ in the physical plane maps into the region L_Θ in the ζ-plane as shown in figure 5. In the transient case the shape of the curve $\widehat{BAF}$ is unknown and must be determined by applying equation (9). However, for steady state equation (9) becomes $|\zeta| = 1$ and this shows that $\widehat{BAF}$ is then a semicircle.

The region Γ in the Ω-plane is chosen to be the semicircular region of unit radius shown in figure 6. Notice that at steady state the mapping $\Omega \rightarrow \zeta$ merely involves multiplication by a negative constant. In the transient situation, however, the mapping function is unknown, but an examination of figures 5 and 6 shows that we may expect this mapping to be continuous on the boundaries of Γ since these are no singularities which can occur there. Hence, the mapping function can be expanded in a Taylor series about the origin which can be expected to converge on the boundaries of Γ. It also follows from figures 5 and 6 that this series must have the form

$$\zeta(\Omega,\Theta) = -K\left(\sqrt{1 - \beta^2}\right) \sum_{n=0}^{\infty} \alpha_n \Omega^{2n+1} \tag{10}$$

where the unknown functions of time β and α_n for $n = 0, 1, 2, \ldots$ are real valued. Elementary mapping techniques can be applied to show that the function which maps Γ into J_Θ in the manner indicated in figures 4 and 6 is defined by:

$$\frac{\partial W}{\partial \Omega} = - \frac{2}{K\left(\sqrt{1 - \beta^2}\right) \sqrt{(1 + \Omega^2)^2 - (1 - \beta^2)(1 - \Omega^2)^2}} \tag{11}$$

Substituting equations (10) and (11) into equation (4) and choosing the origin of the coordinate system in the physical plane to be at point D yields after performing the integration

$$Z(\Omega,\Theta) = -\frac{\beta A_0 - A}{\ln\sqrt{1 - \beta^2}} \ln\left[\frac{\beta\sqrt{X(\Omega)} + (1 + \Omega^2) + (1 - \beta^2)(1 - \Omega^2)}{2\sqrt{1 - \beta^2}}\right] + \sqrt{X(\Omega)} \sum_{n=0}^{\infty} A_n \Omega^{2n} \tag{12}$$

with $X(\Omega) = (1 + \Omega^2)^2 - (1 - \beta^2)(1 - \Omega^2)^2$ and the A_n are related to the α_n by

$$\alpha_n = \sum_{j=0}^{2} \binom{2}{j} [1 - (-1)^j (1 - \beta^2)] \left(n + 1 - \frac{j}{2}\right) A_{n+1-j} + \delta_{n0}\beta\left(\frac{\beta A_0 - A}{\ln\sqrt{1 - \beta^2}}\right) ; n = 0,1,2,\ldots \tag{13}$$

To determine β and the A_n (and hence, the α_n) as functions of time, equations (13), (12), and (10) are inserted into the boundary condition (9). Since $\Omega = e^{i\omega} (0 \leq \omega \leq \pi)$ for $\Omega \in \widehat{FAB}$ the resulting expression involves sines and cosines of ω. To eliminate the ω dependence of this expression it is multiplied through by $\cos(2p\omega)$ for $p = 0, 1, 2, \ldots$ (the restriction to this subset of the complete set of sines and cosines is dictated by symmetry requirements) and integrated between $\omega = 0$ and $\pi/2$. Upon performing these operations we obtain the following infinite set of first order ordinary differential equations which determine β and the A_n as functions of time.

$$\frac{d\beta}{d\Theta}\left[\sum_{k=0}^{\infty} \alpha_k \left(\frac{M}{1 - \beta^2} J^{(1)}_{0,k,p} + \frac{1}{\beta} \sum_{n=0}^{\infty} A_n J^{(2)}_{n,k,p}\right)\right] + \sum_{n=0}^{\infty} \frac{dA_n}{d\Theta} \sum_{k=0}^{\infty} \alpha_k J^{(3)}_{n,k,p}$$

$$= -\pi(1 + \delta_{p0}) \sum_{n=0}^{\infty} \gamma_n \gamma_{n+p} + \frac{2\pi\delta_{p0}}{K\left(\sqrt{1 - \beta^2}\right)}; \quad p = 0, 1, 2, \ldots \tag{14}$$

where

$$\left.\begin{aligned} H^{(0)}_{k,p} &= \frac{2\beta^2}{\ln\sqrt{1 - \beta^2}}\left[\frac{K_{k+p}}{2(k + p) + 1} + \frac{K_{k-p}}{2(k - p) + 1}\right] \\ H^{(1)}_{k,p} &= K_{k+p} + K_{k-p} \end{aligned}\right\} \quad \begin{aligned} p &= 0, 1, 2, \ldots \\ k &= 0, \pm 1, \pm 2, \ldots \end{aligned}$$

and

$$J^{(r)}_{n,k,p} = \sum_{j=0}^{\min[2,r]} \binom{2 - \delta_{r1}}{j} (-1)^j \left[(-1)^{rj} - (1 - \beta^2)\right] H^{(1)}_{(n+j-k-1),p} + \delta_{n0} H^{(0)}_{k,p}$$

$$r = 1, 2, 3 \quad \text{and} \quad n, k, p = 0, 1, 2, \ldots$$

The initial values of β and A_n for these differential equations are chosen so that the desired initial configuration of the frozen region is given by equation (12).

It is not hard to show that the heat flow through the frozen region can be computed from

$$\frac{Q}{2k(t_f - t_w)} = \frac{K(\beta)}{K\sqrt{1 - \beta^2}} \tag{15}$$

once β is known. The shape of the frozen region can be computed from equation (12) with $\Omega = e^{i\omega}$.

RESULTS FOR STEADY STATE

The results for steady state can be obtained as a special case of the preceding analysis by letting the time dependent coefficients A_n be zero and letting β be independent of time. In this case the shape of the solid-liquid interface of the frozen layer is given parametrically by

$$\left.\begin{aligned} X_s^{ss} &= -\frac{A}{\ln\sqrt{1 - \beta^2}} \ln\left(\frac{\beta\cos\omega + \sqrt{1 - \beta^2\sin^2\omega}}{\sqrt{1 - \beta^2}}\right) \\ Y_s^{ss} &= -\frac{A}{\ln\sqrt{1 - \beta^2}} \left\{\omega + \tan^{-1}\left[\frac{-(1 - \beta^2)\sin\omega}{\beta\sqrt{1 - \beta^2\sin^2\omega} + \cos\omega}\right]\right\} \end{aligned}\right\} \begin{aligned} &0 \le \omega \le \pi/2 \\ &-\pi/2 \le \tan^{-1} \le 0 \end{aligned} \tag{16}$$

The boundary condition on the solid-liquid surface now determines β in terms of the physical quantity A. Thus,

$$A = -\frac{\ln\sqrt{1 - \beta^2}}{\beta K\left(\sqrt{1 - \beta^2}\right)} \tag{17}$$

The heat flow through the frozen region is still given by equation (15). For a given A as determined by the imposed temperatures and heat transfer coefficient, β can be found from equation (17). This value of β can then be used in equation (16) to compute the shape of the frozen region and in equation (15) to compute the heat flow through the frozen region.

An analysis similar to the one discussed above shows that at steady state in the case of freezing inside a rectangular duct the shape of the frozen region is given parametrically by

$$\left.\begin{aligned} \frac{X_s^{ss}}{B} &= \frac{A}{B} - \frac{\sqrt{\frac{2}{d+c}}\, F\left[\sin^{-1}\sqrt{\frac{(d+c)(1+\xi)}{(d+1)(c+\xi)}},\ \sqrt{\frac{d+1}{d+c}}\right]}{I(d,c)} \\ \frac{Y_s^{ss}}{B} &= 1 - \frac{\sqrt{\frac{2}{d+c}}\, F\left[\sin^{-1}\sqrt{\frac{(d+c)(1-\xi)}{(1+c)(d-\xi)}},\ \sqrt{\frac{c+1}{d+c}}\right]}{I(d,c)} \end{aligned}\right\};\ -1 < \xi < 1 \quad (18)$$

The constants c and d are given in terms of the physical parameters of the problem by

$$\frac{A}{B} = \frac{I(c,d)}{I(d,c)} = \frac{a}{b}$$

$$\frac{1}{B} = \frac{\frac{2}{\sqrt{(c+1)(d+1)}}\, K\left[\sqrt{\frac{(c-1)(d-1)}{(c+1)(d+1)}}\right]}{I(d,c)} = \frac{k}{hb}\,\frac{t_f - t_w}{t_l - t_f}$$

The heat flow through the frozen region is given by

$$\frac{Q}{4k(t_f - t_w)} = \frac{K\left[\sqrt{\frac{2(c+d)}{(c+1)(d+1)}}\right]}{K\left[\sqrt{\frac{(c-1)(d-1)}{(c+1)(d+1)}}\right]}$$

QUASI-STEADY SOLUTIONS

In most cases a good approximation to the full transient solution discussed above can be obtained by setting all the A_n's equal to zero and letting β be the only unknown function of time. This approximation amounts to assuming that the heat flux is uniform over the solid liquid interface. In this case, the boundary condition on the solid-liquid frozen layer surface implies that β is given as a function of time by

$$\frac{\pi}{2A^2}\Theta = \int_{\beta_{initial}}^{\beta} \frac{k}{1-k^2}\,\frac{K\left(\sqrt{1-k^2}\right)}{\left(\ln\sqrt{1-k^2}\right)^2}\left[\frac{E(k)\ln\sqrt{1-k^2} + k^2K(k)}{kAK\left(\sqrt{1-k^2}\right) + \ln\sqrt{1-k^2}}\right]dk$$

where $\beta_{initial}$ is the value of β at $\Theta = 0$. This approximation to the solution requires that the frozen region pass through a succession of steady states during its transient growth. Thus, the initial configuration must be chosen to be a particular steady state frozen region shape. The value of $\beta_{initial}$ is the value of β determined from the steady state analysis which gives the desired initial steady state shape. Setting $\beta_{initial} = 1$ corresponds to starting the transient when there is no frozen layer on the plate.

RESULTS AND DISCUSSION

In all the full transient solutions which were carried out the initial configuration of the frozen region was taken to be a steady state configuration, hence, the initial conditions on the A_n and β were taken to be $A_n = 0$ and $\beta = \beta_{initial}$ where $\beta_{initial}$ is determined from the steady state analysis in the same way as for the approximate solution discussed above.

A typical set of transient growth curves for the frozen layer forming on a flat plate are shown in figure 7. The results give a qualitative idea of the rate at which solidification occurs. The rate is most rapid at early times when the frozen region has the least thickness, and hence, the least resistance to heat flowing through it. As the frozen region becomes large compared with the width of the plate its shape becomes circular tending toward the axisymmetric solution where the heat removal is through a line sink at the center of the solidified region.

Only the steady state case was considered for freezing inside a rectangular duct. A typical set of steady state contours of the frozen region are shown in figure 8. The curves on this plot are drawn for various values of the controllable physical parameter $B = (hb/k)[(t_l - t_f)/(t_f - t_w)]$. For thin layers B is large and the layer is of constant thickness except close to the corner. As B is decreased (this corresponds to increased cooling) to a value of about 2.5 the frozen layer thickness increases fairly uniformly around the duct. Then as B is decreased by a very small amount, the thickness along the short side increases substantially while that of the long side remains almost constant. For thick frozen regions the interface approaches a circular shape.

One of the interesting aspects of the profiles in figure 8 is that there is a minimum value of B equal to about 2. This phenomena occurs for ducts of all aspect ratios. For smaller values of B (i.e., larger cooling) there are no steady state solutions and the liquid in the duct would freeze completely.

The fact that there is a minimum value of B leads to the most interesting feature of figure 8. For each value of B above the minimum there are mathematically two possible frozen region configurations. It can be shown, however, that the thicker regions are unstable to small disturbances and hence, will not occur physically.

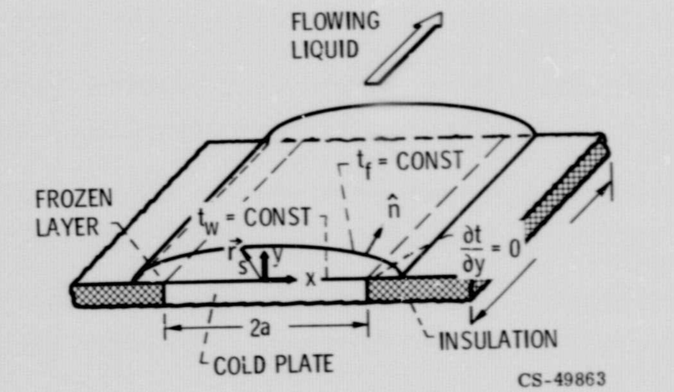

Figure 1. - Two-dimensional solidified layer formed on a cold plate.

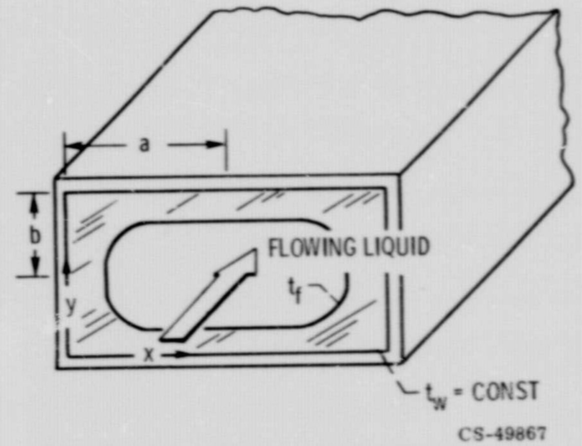

Figure 2. - Cooled rectangular channel containing warm flowing liquid.

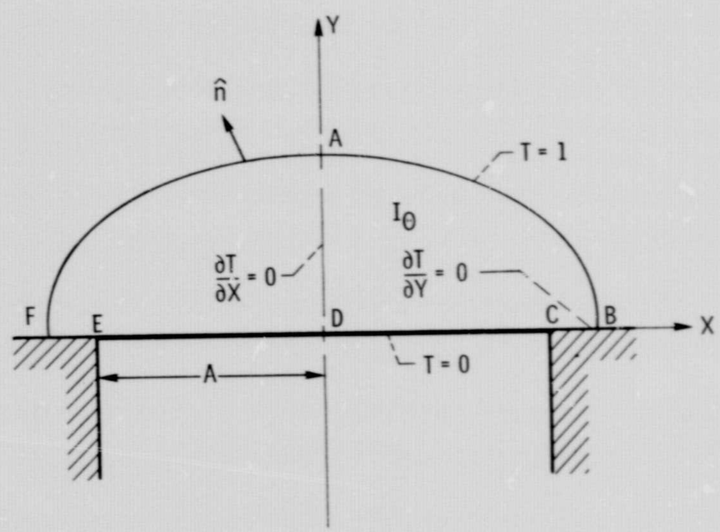

Figure 3. - Dimensionless physical plane, Z = X + iY.

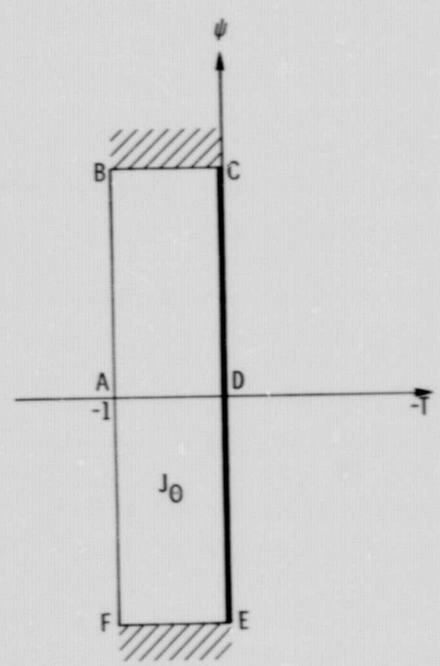

Figure 4. - Potential plane, $W = -T + i\psi$.

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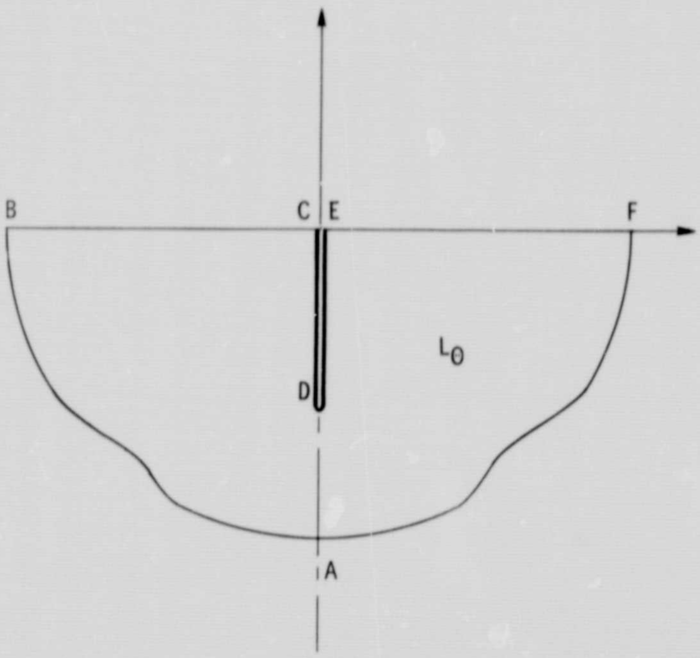

Figure 5. - Temperature derivative plane, $\zeta = \left[-\frac{\partial T}{\partial X} + i\frac{\partial T}{\partial Y}\right]^{-1}$

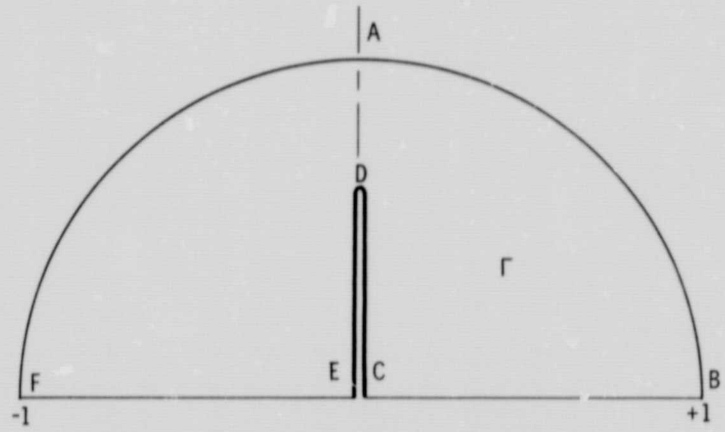

Figure 6. - Intermediate Ω-plane.

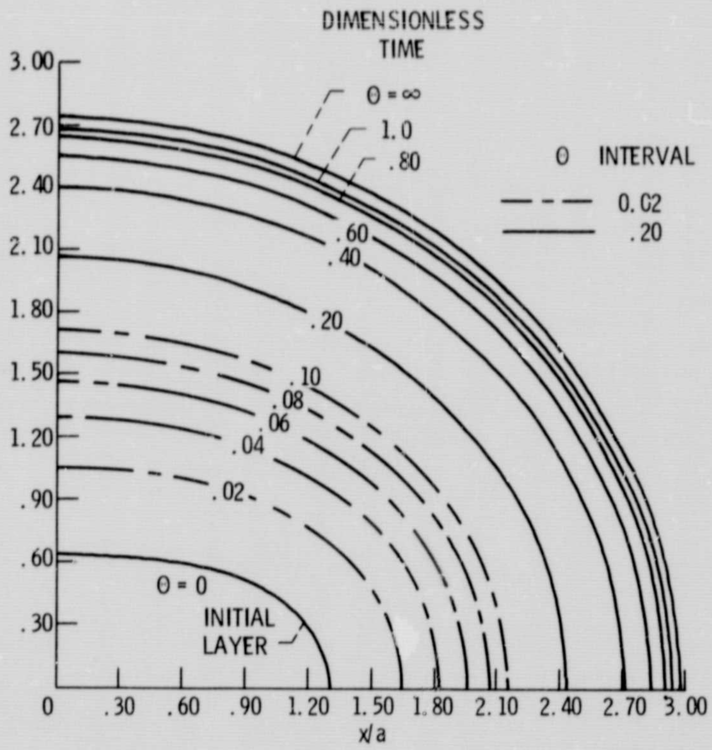

Figure 7. - Transient solidification starting from an initial layer; $\beta_{initial} = 0.995$, A = 0.2.

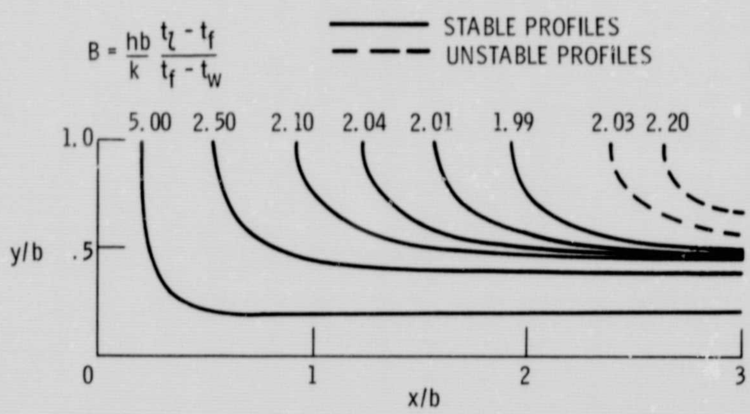

Figure 8. - Steady-state frozen layer profiles; duct aspect ratio, a/b = 3.